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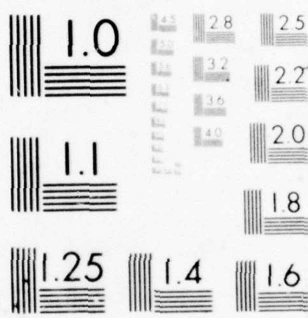

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R. C. Buck

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ABSTRACT

Results are obtained dealing with the exact and the approximate representation of a function F as a superposition, in designated formats, of functions of fewer variables. Two main cases are considered. In the classical nomographic case, discussed in Sections 5, 6, 7, one seeks criteria for deciding if a function can be expressed in the form $f(\phi(x)+\psi(y))$, or as a uniform limit of such functions. The second case, discussed in Sections 2, 3, 4, is also related to the solution of Hilbert's 13th problem, and deals with the format $F(x) = f(\phi(x))$ where x lies in an n-cell I and ϕ is a real valued continuous function on I, and f is a function on R taking values in a chosen normed space $\mathcal{E}$. The use of these criteria is illustrated with several specific functions.

Since each format is associated with a specific partial differential equation, the results raise questions about the nature of the uniform closure of the C^{∞} solutions of such equations. Section 3 may also have more general interest since it shows that every continuous real function on an n-cell must share a certain universal property related to the metric dispersion of its level sets.

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APPROXIMATE COMPLEXITY AND FUNCTIONAL REPRESENTATION

R. C. Buck

1. Summary

In the last four decades, there has been a growing interest in the theory of functional complexity, focussed initially on the problem of representing a given function as a superposition of functions of fewer variables, and stemming from the challenge of Hilbert's 13th problem. Not to be overlooked, however, are the early papers of Nina Bary dealing with representability of measurable and continuous functions [4] and the persistent interest in Russia and elsewhere in nomography and the use of mechanical linkages to generate specific functions. [15] This has now taken on additional interest with the advent of the computer and the introduction of new freedoms and new constraints; for example, composition is a faster operation than multiplication.

Many of the basic questions have been answered by the works of Vitushkin, Arnol'd and Kolmogoroff, [23] [1] [13]. In particular, the former has shown that the number n/p is a convenient index to measure the total complexity of the class of functions of n variables that have continuous derivatives of orders up to and including p. By a category argument, one can show that there are functions not representable by composition of functions of lower complexity. (See the excellent account of this in Lorentz [14]). If only continuity is required, corresponding to $p = 0$, the Vitushkin index is no longer useful. One basic question,

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posed in a special case by Hilbert, is answered by the Kolmogoroff result, showing that every continuous function of n variables can be expressed in terms of continuous functions of one variable, and the single binary function + . A very accessible proof is found in Hedberg [18]. (See also [3] [19] [24]).

There are two directions to the present paper. The first has to do with the development of criteria for deciding if specific functions are non-representable in assigned formats. The second has to do with approximate complexity; clearly, a function that can be uniformly approximated by functions of low complexity should also be regarded as having low complexity.

Most of the space is devoted to two cases, the classical nomographic case in which one looks at $F(x, y) = f(\phi(x) + \psi(y))$, and one similar to that examined by Arnol'd, in which $F(x) = f(\phi(x))$, where ϕ is a real valued function on an n-cell and the values of f and F may be in any normed space $\mathcal{E}$. The latter is discussed in Sections 2, 3 and 4, and the former in Sections 5, 6 and 7.

There is also close contact with some of the work of Doss [8] and Sprecher [20] [21] who also have made use of the level sets of continous functions in their study of functional representation. Section 3, dealing with metric dispersion properties of level sets, may have much more general interest.

Finally, we have stated a number of conjectures to which we have been led by this study. Some are connected with the behavior of the partial differential equations that are characteristic of the function classes considered; while it is conceivable that a very smooth function F might be representable in a given format using continuous unsmooth component functions, we conjecture that F must *itself* satisfy the associated differential equation. This is confirmed by some of the results in Sections 2 and 4 for polynomials, using facts about the zeros of polynomials in several variables.

Some of these results have been announced in [6].

2. The schema $f(\phi(x, y), z)$

This case, which has been studied elsewhere ([1], [16]) is a suitable introduction to representability problems. We first set the notation. Let $\mathfrak{G}$ be an open connected set in R^3. We write $\mathfrak{F}_n(\mathfrak{G})$ for the class of real valued functions that are defined in $\mathfrak{G}$ and have there a representation of the form

$$F(x, y, z) = f(\phi(x, y), z) \tag{1}$$

where f and ϕ are of class C^n on appropriate open sets in R^2. Thus, $\mathfrak{F}_0$ will be those functions for which only continuity is required, and $\mathfrak{F}_0 \supset \mathfrak{F}_1 \supset \mathfrak{F}_2$.

Historically, an important step toward the eventual solution of Hilbert's 13th was Arnol'd's discovery that the convex hull of $\mathfrak{F}_0$ contained all continuous functions of three variables [1]. It had earlier been observed by Pólya that if f and ϕ were completely unrestricted, then the class of functions represented by the format (1) was in fact universal; every function

$F(x, y, z)$, real valued or not, has such a representation. [16]. One need only choose ϕ as a bijection, mapping R^2 onto R. Most of our results will deal with the intermediate class $\mathfrak{F}_w$ which properly contains $\mathfrak{F}_0$, and consists of functions of the form (1) where ϕ is required to be continuous but f is entirely unrestricted.

In contrast with the facts mentioned above, the class of smoothly represented functions, those in $\mathfrak{F}_n$ for some $n \geqq 1$, comprise a very thin subset of the class of all continuous functions. The proof of the following is routine.

Theorem 1. If $F \varepsilon \mathfrak{F}_2(\mathfrak{G})$, then in $\mathfrak{G}$, F satisfies the differential equation

$$F_x F_{yz} - F_y F_{xz} = 0 . \tag{2}$$

Conversely, if F satisfies (2) in $\mathfrak{G}$, then it is locally of the format (1) in $\mathfrak{G} - \Gamma$, where Γ is the set where $F_x F_y = 0$.

Corollary: Any function in $\mathfrak{F}_0(\mathfrak{G})$ can be approximated locally, uniformly on compact sets, by functions in $\mathfrak{F}_\infty$ which satisfy equation (2).

We next seek a weaker characterization theorem that does require us to deal with functions that are twice differentiable. As a first step, observe that Theorem 1 can be restated. For fixed z, $z = c$, consider the planar mapping T defined by

$$T: \begin{cases} u = F(x, y, c) \\ v = F_z(x, y, c) \end{cases} \tag{3}$$

Then, equation (2) is the same as $\partial(u, v)/\partial(x, y) = 0$, and thus the criterion given in Theorem 1 for F to belong to $\mathfrak{F}_2$ becomes the statement that the mapping T is everywhere locally singular, for each choice of c.

This observation leads in turn to a C' characterization. Given a function $F(x, y, z)$ of class C' on $\mathfrak{G}$, which for simplicity we take to be an open box $X \times Y \times Z$, take c_1 and c_2 in Z and then define a planar mapping T by

$$T: \begin{cases} u = F(x, y, c_1) \\ v = F(x, y, c_2) \end{cases}$$

Theorem 2. *If $F \in \mathfrak{F}_1(\mathfrak{G})$, then the mapping T given by (4) is locally singular in $X \times Y$ for every choice of c_1 and c_2 in Z in the sense that $\partial(u, v)/\partial(x, y) = 0$. Conversely, if T is locally singular in $X \times Y$ for all choices of c_1, c_2, then the function F is locally representable in the form (1) in the set $\mathfrak{G} - \Gamma$.*

To prove the second half, let (x_0, y_0, z_0) be a point where $F_x F_y \neq 0$. The hypothesis on T then implies that there is a function $\beta(x, y)$, defined and non-zero on a neighborhood of $p_0 = (x_0, y_0)$ such that $F_x(x, y, z) - \beta(x, y) F_y(x, y, z) = 0$ for all z near z_0 and all (x, y) near p_0. Let $\phi(x, y)$ be a solution of the equation $\phi_x - \beta\phi_y = 0$ near p_0. Then, one finds that there exists a function f such that $F(x, y, z) = f(\phi(x, y), z)$ for all z near z_0 and all (x, y) near p_0.

This elementary result is sufficient to show that specific functions (e.g. $xy + yz + xz$ or $x^2y + y^2z + z^2x$) do not belong to the class $\mathfrak{F}_1$ locally. However, this does not mean that such functions could not belong to either of the larger classes $\mathfrak{F}_0$ or $\mathfrak{F}_w$. Although each function mentioned is in C^∞, it might possibly be representable in the form (1) with non-smooth choices for f and ϕ. The next result, which is a partial characterization of these larger classes, allows one to settle this question, and in turn can be extended to handle the uniform closure of either $\mathfrak{F}_0$ or $\mathfrak{F}_w$.

Before proceeding, it is helpful to recast the original problem. Instead of considering a function F from $X \times Y \times Z$ into R, it is equivalent to regard F as a mapping from $X \times Y$ into the function space $C[Z]$, which may as well be replaced now by any normed space $\mathcal{E}$. The problem of representing F in the format $F(x, y, z) = f(\phi(x, y), z)$ is replaced by the simpler format $F = f \circ \phi$, and becomes a familiar factoring problem. Given F, a mapping from $X \times Y$ into $\mathcal{E}$, do there exist a real valued function ϕ and an $\mathcal{E}$-valued function f for which the following diagram commutes?

$$\text{(5)}\qquad \begin{tikzcd} X \times Y \arrow[r, "\phi"'] \arrow[dr, "F"'] & R \arrow[d, "f"] \\ & \mathcal{E} \end{tikzcd}$$

If no restrictions are placed on ϕ or f, then the answer is affirmative for arbitrary F, merely by taking ϕ as a bijection from $X \times Y$ into R. However, if ϕ is required to be continuous, then this is not the case. We say that a sublevel set for a function is one on which the function is constant. Clearly, from (5), any set E that is a sublevel set for ϕ must also be one for F. Thus, any universal property of sublevel sets of real value continuous functions on $X \times Y$ will also hold for certain sublevel sets of any representable F.

Lemma 1. Let ϕ be any real valued continuous function on an n-cell S, $n \geqq 2$. Take any $p_0 \in S$. Then, either ϕ is locally constant at p_0, or every neighborhood of p_0 contains a non-countable collection of distinct sublevel sets, each of which is non-countable.

For, if θ is a convex neighborhood of p_0 on which ϕ is not constant, we may choose p_1, p_2 in θ with $\phi(p_1) \neq \phi(p_2)$ and any number c with $\phi(p_1) < c < \phi(p_2)$; then, every arc in θ from p_1 to p_2 contains a point on the c-level set for ϕ.

Applied to the representability problem, this yields the following simple criterion; note that it deals with the weak class $\mathfrak{F}_w$ in which no restriction is placed on the function f.

Theorem 3. Let F be any mapping from an n-cell S $(n \geqq 2)$ into $\mathcal{C}$ which is of the form $F = f \circ \phi$, where ϕ is continuous from S into R. Then, F must be locally singular in S, meaning that locally, either F is constant, or F has a non-countable number of distinct non-countable sublevel sets.

We note that this result is only effective if $\mathcal{C}$ is larger than R itself, so that there exist functions F on S to $\mathcal{C}$ which do not share this property of the representable functions. For example, applied to the original classical problem, we see at once that $xy + yz + zx$ and $x^2y + y^2z + z^2x$ do not belong to the class $\mathfrak{F}_0$, or even to the weak class $\mathfrak{F}_w$, on any open set since the associated mapping F from $X \times Y$ into $C[Z]$ is easily seen to be at most 2-to-1, and therefore does not have any large sub-level sets, as required by Theorem 3. In essence, this simple argument is much the same as that used by Pólya in [16].

Study of these and similar examples leads to the following conjecture, for which only incomplete evidence has been obtained: _if_ F _is a polynomial that belongs to the weak class_ $\mathfrak{F}_w$ _on an open set_ $\mathfrak{G}$, _then_ F _must satisfy on_ $\mathfrak{G}$ _the differential equation_ (2).

Before turning to the problem of approximate representability, and the search for criteria that must be satisfied by any function that is the uniform limit of functions in $\mathfrak{F}_w$, or in $\mathfrak{F}_0$, we must obtain more refined results dealing with the nature of the level sets of real valued functions ϕ defined on an n-cell with $n \geqq 2$.

3. Dispersion properties of level sets

Let ϕ be a real valued function defined on an n-cell S. For any real number λ, the λ-level set for ϕ is the set E_λ of all points $p \in S$ with $\phi(p) = \lambda$, i.e. $\phi^{-1}(\lambda)$. If ϕ is not required to be continuous, then each of these sets E can be finite, and in fact a solitary point. If $n \geqq 2$ and ϕ is continuous, we have seen (Lemma 1) that infinitely many of these sets E must be infinite. In the present section, we show that there must always be one of these level sets that achieves a certain minimal dispersion in S, independently of the choice of ϕ. The measure of dispersion or size which we use is a familiar one related to the notion of metric capacity of subsets [14]. Given a set E and a number $\delta > 0$, we look for points $p_i \in E$ that are mutually separated by δ, so that $|p_i - p_j| \geqq \delta$ for $i \neq j$. Then, $\underset{\sim}{n}(E, \delta)$ is the maximum number of such points that can be obtained. If $E \subset S$, then $\underset{\sim}{n}(E, \delta) \leqq n(S, \delta)$, and the comparative size of these measures the degree to which the set E is dispersed in the set S. If S is an n-cell in R^n of side L, then $\underset{\sim}{n}(S, \delta) \approx (L/\delta)^n$, as $\delta \downarrow 0$.

Our main result in this direction is the following.

Theorem 4. Let S be an n-cell of side L, and ϕ any real valued continuous function defined on S. Then, for any $\delta > 0$,

(i) if $n = 2$ and $\delta < L/2$, ϕ must have a level set E

(6) for which $\underset{\sim}{n}(E, \delta) \geqq L/\delta$

(ii) If $n \geqq 3$ and $\delta < L/(16)$, then ϕ must have a level set E for which

(7) $$n(E, \delta) > \frac{1}{2^{n-1}} (L/\delta)^{n/2}$$

The conclusion given in (i) is easily seen to be best possible; this does not seem likely for part (ii), and we conjecture that the exponent $n/2$ can be replaced by $n - 1$. More generally, we conjecture that the following is true: if ϕ is a continuous mapping from a compact metric space A onto a metric space B, then there is a constant C depending only on A and B such that ϕ must have a level set $E \subset A$ for which $\underset{\sim}{n}(E, \delta)\ \underset{\sim}{n}(B, \delta) \geq C\ \underset{\sim}{n}(A, \delta)$, for all sufficiently small δ.

While the proofs for part (i) and (ii) are similar, we give them separately since the conclusions are different. Our first proof for part (i) was improved by a suggestion made by Carl de Boor.

Proof of (i). Let S be a square of side L, and $0 < \delta < L/2$. Choose $k = [L/\delta]$ and consider $k+1$ evenly spaced vertical segments $\alpha_0, \alpha_1, \ldots, \alpha_k$ in S, each of length L. Let σ_i be the real interval $\phi(\alpha_i)$, and suppose that $\cap\, \sigma_i$ is non empty. If λ lies in each σ_i, then we can choose $p_i \varepsilon\, \alpha_i$ so that $\phi(p_i) = \lambda$, and we have found $k+1$ points in the λ-level set for ϕ which are mutually separated by δ. Thus, $\underset{\sim}{n}(E_\lambda, \delta) \geq k+1 > L/\delta$. If $\cap\, \sigma_i$ is empty, then some pair of intervals, say σ_i and σ_j, are disjoint. Choose λ so that $\phi(q_1) < \lambda < \phi(q_2)$ for all q_1 in α_i and q_2 in α_j. Construct $k+1$ horizontal segments $\beta_0, \beta_1, \ldots, \beta_k$ in S evenly spaced and each intersecting the vertical segments α_i and α_j (See Figure 1). Choose $p_i \in \beta_i$ so that $\phi(p_i) = \lambda$. These again are points in the λ-level set E_λ and are mutually separated by δ, so that again $\underset{\sim}{n}(E_\lambda, \delta) > L/\delta$.

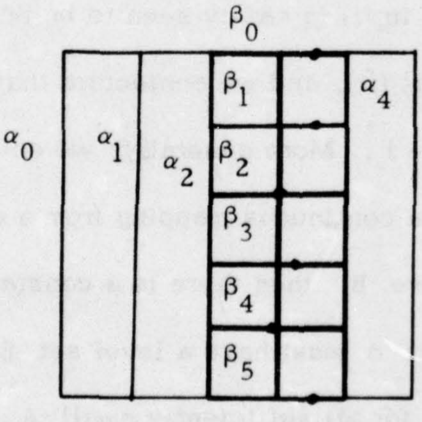

Figure 1

Proof of (ii). Let S be an n-cell of side L with $n \geq 3$, and suppose that $0 < \delta < L/(16)$. Choose three integers as follows:

$$k = [L/\delta]$$

(8) $$m = [(L/\delta)^r] \quad \text{where } r = \frac{n-2}{2n-2}$$

(9) $$p = [L/(2m\delta)] \quad .$$

(Thus, when $n = 3$, $m^4 \approx L/\delta$ and $p^4 \approx (L/\delta)^3$.)

Consider $k+1$ horizontal sheets in S, each an $(n-1)$ cell of side L, parallel and uniformly spaced at distance $L/k > \delta$. Divide each sheet into m^{n-1} small $(n-1)$ cells of side L/m . Shrink each of these by a factor of $1/2$, leaving its center fixed. Each sheet will then contain a collection of m^{n-1} disjoint $(n-1)$ cells of side $L/(2m)$, and mutually separated by $L/(2m)$. (Figure 2 shows one such sheet, for $n = 3$, $m = 4$.)

It is easily checked that $L/(2m) > \delta$, using the fact that $L/\delta > 16 > 2^{2(1-2/n)}$. We now have, on all the sheets, a collection of $(k+1)m^{n-1}$ small $(n-1)$ cells α_i, each of side $L/(2m)$ and such that any pair of points from distinct α_i are separated by at least δ . Let ϕ be a continuous function defined on S, and let $\sigma_i = \phi(\alpha_i)$. Suppose first that $\bigcap \sigma_i$ is not empty. Choose λ so that $\lambda \in \sigma_i$ for all i, and then $p_i \in \alpha_i$ with $\phi(p_i) = \lambda$. We have therefore found $(k+1)m^{n-1}$ points in the λ-level set of ϕ with mutual distance δ . Accordingly,

$$\underset{\sim}{n}(E_\lambda, \delta) \geq (k+1)m^{n-1} \quad .$$

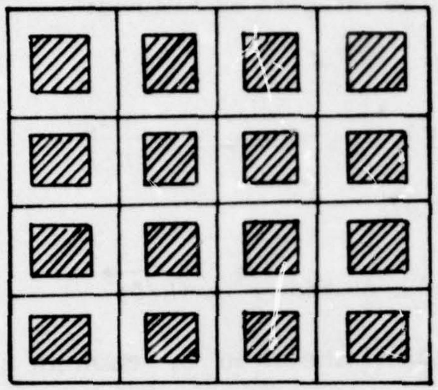

Figure 2

However, from (8),

$$m > (L/\delta)^r - 1 > \frac{1}{2}(L/\delta)^r$$

since $L/\delta > 16 > 2^{1/r}$.

Thus, we have shown that

$$\underset{\sim}{n}(E_\lambda, \delta) > (k+1)\left\{\frac{1}{2}(L/\delta)^r\right\}^{n-1}$$

$$> \left(\frac{L}{\delta}\right)\frac{1}{2^{n-1}}\left(\frac{L}{\delta}\right)^{(n-1)r}$$

$$> \frac{1}{2^{n-1}}(L/\delta)^{n/2} .$$

Suppose now that the intersection of all the real intervals σ_i is empty. By Helly's theorem, two intervals, say σ_i and σ_j, must be disjoint. Choose a real number λ so that

(10) $$\phi(q_1) < \lambda < \phi(q_2)$$

for all q_1 in α_i and q_2 in α_j. Look at α_i which is an $(n-1)$ cell of side $L/(2m)$, and divide it by a mesh of width $(L/2m)/p = L/(2mp)$ to produce $(p+1)^{n-1}$ points q_i with each pair separated by at least $L/(2mp)$. Observe that, by (9), $L/(2mp) > \delta$. Do the same for the second $(n-1)$ cell α_j, producing points q_i^* and then join q_i to q_i^* by an arc β_i in S so that no two points on distince arcs are closer than δ apart. Because of (10), we can now choose a point p_i on each arc β_i with $\phi(p_i) = \lambda$. We then have $(p+1)^{n-1}$ points in the λ-level set of ϕ, mutually separated by δ, and

$$\underset{\sim}{n}(E_\lambda, \delta) \geq (p+1)^{n-1}$$

which by (8) and (9) yields

$$\underset{\sim}{n}(E_\lambda, \delta) \geq \left(\frac{L}{2m\delta}\right)^{n-1} = \frac{1}{2^{n-1}}\left(\frac{L}{\delta}\right)^{n-1}\frac{1}{m^{n-1}}$$

$$\geq \frac{1}{2^{n-1}}(L/\delta)^{r(n-1)}(\delta/L)^{r(n-1)}$$

$$\geq \frac{1}{2^{n-1}}(L/\delta)^{n-1}(\delta/L)^{(n-2)/2}$$

$$\geq \frac{1}{2^{n-1}}(L/\delta)^{n/2}$$

completing the proof of part (ii) of Theorem 4.

We note that if the more general conjecture given just below the statement of the theorem were true, then the exponent $n/2$ would be replaced by $n-1$ since $\underset{\sim}{n}(S, \delta) \approx (L/\delta)^n$ and $\phi(S)$ is an interval J for which $\underset{\sim}{n}(J, \delta) = O(1/\delta)$. We have not been able to verify this, nor to obtain a theorem similar to Theorem 4 for continuous mappings from an n-cell into R^s with $2 \leqq s \leqq n-1$. In this case, the conjecture is that $\underset{\sim}{n}(E, \delta) \approx (L/\delta)^{n-s}$.

4. The closure of $\mathfrak{F}_W$

We now apply the results of the preceding section to obtain criteria for approximate representability in the class $\mathfrak{F}_W$. Let S be an n-cell of side L, and $\mathcal{E}$ be a normed space, and let $\mathfrak{F}_W$ be the class of mappings F from S to $\mathcal{E}$ which have the form $F(p) = f(\phi(p))$, where ϕ is a continuous real valued function on S, and f is an arbitrary function from R to $\mathcal{E}$.

Theorem 5. If G is a function on S to $\mathcal{E}$ that can be uniformly approximated on S by functions in $\mathfrak{F}_W$, then, G must have level sets that have arbitrarily large finite cardinal. Indeed, given $0 < \delta < L/(16)$, there must exist a level set E for G such that

$$\underset{\sim}{n}(E,\delta) > \frac{1}{2^n}(L/\delta)^{n/2} . \tag{11}$$

Proof: Given $\epsilon > 0$, suppose that there exists $F \varepsilon \mathfrak{F}_W$ with $\|F - G\| < \epsilon$. Write F as $f \circ \phi$. By Theorem 4, we may choose a level set E for ϕ such that E obeys (7). If $p \varepsilon E$, then $F(p) = f(\lambda)$, independent of p, so that E is also a level set for F. Since $|F(p) - G(p)| < \epsilon$ for all $p \varepsilon S$, it follows that E is an approximate level set for G; indeed, if N is $[2^{-n}(L/\delta)^{n/2}]$, then for each $\epsilon > 0$ we can choose N points p_i in S such that $|p_i - p_j| \geq \delta$ for $i \neq j$, while $|G(p_i) - G(p_j)| < 2\epsilon$. Letting ϵ decrease, and using the compactness of S, we can arrive at

N points $\{p_i^*\}$ in S with $|p_i^* - p_j^*| \geq \delta$ while $G(p_1^*) = G(p_2^*) = \ldots = G(p_N^*)$, and we have found a level set for G obeying (11). Since δ can be arbitrarily small, G must have level sets with arbitrarily many points.

Applied to specific cases, this result shows immediately that the test functions $xy + yz + zx$ and $x^2y + y^2z + z^2x$ cannot be approximated uniformly by functions of the form $f(\phi(x, y), z)$ on any open set in R^3 since they are at most 2-to-1 as mappings from R^2 into $C[R]$. More generally, the criterion in Theorem 5 implies that any function in the uniform closure of $\mathfrak{F}_W$ must be locally singular; every neighborhood must contain arbitrarily large finite sets on which the function is constant. Accordingly, G will not lie in the closure of $\mathfrak{F}_W$ if the sets $G^{-1}G(p)$ are uniformly finite. This fact, together with some properties of the real zeros sets of polynomials, suggests the following conjecture: *no polynomial* $G(x, y, z)$ *lies in the closure of* $\mathfrak{F}_W(\mathfrak{G})$ *for an open set* $\mathfrak{G}$ *unless* G *satisfies in* $\mathfrak{G}$ *the equation* (2).

The technique used above permits one to use Theorem 4 to estimate the uniform distance from specific functions G to $\mathfrak{F}_W$. We first sketch the general approach, and then illustrate it with one of the test functions. Suppose that G is a function on the square I^2 into $\mathcal{C}$ such that $G^{-1}G(p)$ never contains more than m points. If G is continuous, then for any q and $\epsilon > 0$, there is a δ such that if $|G(p) - G(q)| < \delta$, then p is within ϵ of one of the points in $G^{-1}G(q)$. Suppose that we can establish a uniform quantitative version of this with the following form:

(12) If $|G(p) - G(q)| < d$, then p is within a distance $D = D(d)$ of $G^{-1}G(q)$, for all q, where D is a monotone function of d and $D(0) = 0$.

Assuming that such a result has been obtained, we can estimate the distance from G to $\mathfrak{F}_W(I^2)$. Let $\|G - F\| < \epsilon$ for some $F \varepsilon \mathfrak{F}_W$. Using Theorem 4, choose $m+1$ points p_i in I^2 such that $|p_i - p_j| \geqq L/m$ for $i \neq j$, and the p_i also belong to a level set for F. We will then have $|G(p_i) - G(p_1)| < 2\epsilon$ for all i. Use (12) with $d = 2\epsilon$, $p = p_i$, and $q = p_1$, and we see that p_i will lie within D of the set $G^{-1}G(p_1)$ for each i. However, this set contains at most m points, so that there must exist two points p_i and p_j with $i \neq j$ such that $|p_i - p_j| < 2D$. Accordingly, $2D(2\epsilon) > L/m$ and $\epsilon > (1/2)D^{-1}(L/2m)$, giving the desired estimate for the distance from G to $\mathfrak{F}_W$.

To show how this process works in practice, we apply it to the function $G(x, y, z) = xy + yz + zx$, regarded as a mapping from a square I^2 into $C[R]$; we take $I = [c, c^*]$ where $c^* = c + L$, and $c > 0$. The needed step is a version of the required uniform inverse theorem (12). It is helpful to note that in the present case,

$$G^{-1}\, G(a, b) = \{(a, b), (b, a)\}.$$

<u>Lemma</u> 2. <u>Let</u> (x, y) <u>and</u> (a, b) <u>be points in the square</u> I^2, <u>with</u> $|G(x, y) - G(a, b)| < d$. <u>Then,</u>

$$|(x, y) - (a, b)|\,|(x, y) - (b, a)| < 6\sqrt{2}\,(c^*/L)d.$$

Proof: Since

$$|G(x,y) - G(a,b)| = \max_{c \leqq z \leqq c^*} |xy - ab + (x+y-a-b)z|$$

we have

$$|xy - ab + (x+y-a-b)c| < d$$

$$|xy - ab + (x+y-a-b)c^*| < d$$

which imply

$$|x+y-a-b| < 2d/L$$

(13) $$|xy - ab| < (c + c^*)(d/L) .$$

Now,

$$|(x-y+a-b)(x-y+b-a)| \leqq$$

$$|x+y-a-b|\,|x+y+a+b| + 4|xy-ab|$$

$$\leqq (2d/L)(4c^*) + 4(c+c^*)(d/L) < 16c^*(d/L) .$$

Now,

$$2|(x,y) - (a,b)|^2 = |x+y-a-b|^2 + |x-y+b-a|^2$$

$$2|(x,y) - (b,a)|^2 = |x+y-a-b|^2 + |x-y+a-b|^2 .$$

Setting $|x+y-a-b| = w$, $p = (x,y)$, $q = (a,b)$ and $q^* = (b,a)$ we have

$$(|p-q|^2 - w^2/2)(|p-q^*|^2 - w^2/2) \leqq$$

$$\frac{1}{4}|x-y+b-a|^2|x-y+a-b|^2 < 64(c^*d/L)^2 .$$

Accordingly,

$$|p-q|^2|p-q^*|^2 < 64(\frac{c^*d}{L})^2 + \frac{w^2}{2}\{|p-q|^2 + |p-q^*|^2\}$$

and using (13) and the fact that I^2 has diameter $\sqrt{2}\ L$,

$$|p - q|^2 |p - q^*|^2 < \frac{d^2}{L^2} \{64 (c^*)^2 + 8 L^2\}$$
$$< 72(c^* d/L)^2$$

from which the conclusion follows.

Following the general outline given earlier, we may now use this to prove the following estimation theorem.

<u>Theorem</u> 6. <u>The function</u> $G(x, y, z) = xy + yz + zx$ <u>in</u> $C[I^3]$, <u>where</u> $I = [c, c+L]$ <u>and</u> $c > 0$, <u>is separated from the set</u> $\mathfrak{F}_w(I^3)$ <u>by at least the distance</u> $L^3(c+L)^{-1}/(96\sqrt{2})$.

Proof: Suppose that $F \in \mathfrak{F}_w$ with $\|G - F\| < \epsilon$. Choose three points p_i in I^2 with $|p_i - p_j| \geqq L/2$ for $i \neq j$ but with $F(p_1) = F(p_2) = F(p_3)$. We have $|G(p_i) - G(p_j)| < 2\epsilon$ and, by Lemma 2 with $d = 2\epsilon$,

$$|p_i - p_j||p_i - p_j^*| < 6\sqrt{2}\ (c^*/L)(2\epsilon)$$

where $c^* = c + L$ and $(a, b)^* = (b, a)$. Since $|p_i - p_j| \geqq L/2$ we may conclude that for $i \neq j$,

$$|p_i - p_j^*| < \frac{24\sqrt{2}\ c^*}{L^2}\ \epsilon\ .$$

Using this for $j = 2$ and for $i = 1$ and $i = 3$, we have

$$\frac{L}{2} \leqq |p_1 - p_3| \leqq |p_1 - p_2^*| + |p_3 - p_2^*| < \frac{48\sqrt{2}\ c^*}{L^2}\ \epsilon$$

from which we obtain $\epsilon > (96\sqrt{2})^{-1} L^3/c^*$ as stated.

While this estimate is unlikely to be sharp, it decreases with L in about the way one would expect. It would be of interest to obtain general results of the form (12) to replace the type of ad hoc argument given in Lemma 2. [That this is not possible for all continuous functions F is shown by the trivial example $F(x) = x^3 - x^2$.]

5. The class of nomographic functions

One of the classical questions of functional representability is whether a given function $F(x, y)$ can be given as a simple three scale nomogram; specifically, are there three functions of one variable f, ϕ, ψ, such that

$$F(x, y) = f(\phi(x) + \psi(y)) \tag{14}$$

such questions, together with the strong interest in nomography promoted by d'Ocagne [15], gave rise to the original formulation by Hilbert of the 13th problem, and the solution by Arnol'd and Kolmogoroff. [1] [13].

We continue to use the notation $\mathfrak{F}_n$ for the class of functions with a specific composition format, but this time referring to that given in (14). As before, $\mathfrak{F}_0$ are those that are representable with continuous f, ϕ, ψ, and $\mathfrak{F}_w$ now refers to those for which ϕ and ψ are continuous, but f is unrestricted. Those functions that are smoothly represented form a very thin subset of the continuous functions.

<u>Theorem</u> 7. <u>If</u> $F \in \mathfrak{F}_3(\mathfrak{G})$, <u>then in</u> $\mathfrak{G}$, F <u>obeys the differential equation</u>

$$(F_x F_y)(F_x F_{xyy} - F_y F_{xxy}) + F_{xy}(F_y^2 F_{xx} - F_x^2 F_{yy}) = 0. \tag{15}$$

<u>Conversely, any solution of</u> (15) <u>in</u> $\mathfrak{G}$ <u>is of the form</u> (14) <u>locally in</u> $\mathfrak{G} - \Gamma$, <u>where</u> Γ <u>is the set where</u> $F_x F_y = 0$.

The proof is routine. For the converse, if F obeys (15) then in $\mathfrak{G} - \Gamma$, we have

$$\frac{\partial}{\partial y}\left(\frac{F_y F_{xx} - F_x F_{yy}}{F_x F_y}\right) = 0 .$$

Locally, we can then choose a function $\phi(x)$, with $\phi'(x) \neq 0$ such that

$$(F_y F_{xx} - F_x F_{yy})\, \phi'(x) = F_x F_y \phi''(x)$$

and have

$$\frac{\partial}{\partial x}\left(\frac{F_y}{F_x}\, \phi'(x)\right) = 0 .$$

Locally, we can then choose $\psi(y)$ and have

$$\psi'(y)\, F_x - \phi'(x)\, F_y = 0$$

and neither ϕ' nor ψ' will vanish at any point in $\theta - \Gamma$. Put $u = \phi(x) + \psi(y)$, $V = \phi(x) - \psi(y)$ and define G by $G(u,v) = F(x,y)$, locally. Since $G_v = 0$, $F(x,y) = G(u,-) = f(u) = f(\phi(x) + \psi(y))$, as required.

Corollary: Any function F in $\mathfrak{F}_0(\mathfrak{G})$ can be approximated locally, uniformly on compact sets, by functions in $\mathfrak{F}_\infty$ which satisfy the equation (15).

The result in Theorem 7 says little about the problem of nomographic representability in the classes $\mathfrak{F}_n$ for $n \leqq 2$ or in the weak class $\mathfrak{F}_w$. As in the problem studied in Section 2, it is tempting to hope that if F itself is sufficiently smooth and belongs to the class $\mathfrak{F}_w$ on an open set $\mathfrak{G}$ then F will satisfy (15) in $\mathfrak{G}$.

It is again helpful to recast the situation as a factoring problem. Given F, we ask if there are functions h and f such that the following diagram commutes

$$(16)\qquad \begin{tikzcd} R^2 \supset S \arrow[r, "h"] \arrow[dr, "F"'] & R \arrow[d, "f"] \\ & R \end{tikzcd}$$

Here, f is unrestricted, and h belongs to the class $\mathscr{K}$ of continuous functions of the form $h(x, y) = \phi(x) + \psi(y)$.

Of course, if h could be chosen to be a bijection, $R^2 \to R$, every function F is nomographic. Such a choice for h could not be continuous on R^2; however, it is possible to construct a bijection h whose component functions ϕ and ψ are each continuous off a countable (dense) set.

Theorem 8. Any function $F(x, y)$ can be represented in the form $f(\phi(x) + \psi(y))$. More generally, any function F on the unit n-cell $I^n = [0,1]^n$ can be written in the form

$$F(x_1, x_2, \ldots, x_n) =$$
$$f(\phi(x_1) + 2\phi(x_2) + 4\phi(x_3) + \ldots + 2^{n-1}\phi(x_n))$$

where ϕ is an increasing function on $[0,1]$ to $[0,1]$.

Proof: The key to the construction of h is the observation that each integer between 0 and $2^n - 1$ has a unique expression as a sum of powers of 2; the resulting function h will then be a bijection from $[0,1]^n$ into R, and given F we define f as $F \circ h^{-1}$ and have $F = f \circ h$. The case $n = 3$ is typical. Let t be any real number, $0 \leqq t < 1$, and write t in binary form as

$$t = .t_1 t_2 t_3 \ldots = \sum_1^\infty 2^{-j} t_j$$

where each t_j is either 0 or 1. To obtain uniqueness, we replace any terminal string of 1s by an equivalent expression with terminal 0s. We then define an increasing function ϕ by

$$\phi(t) = \sum_1^\infty 8^{-j} t_j = .t_1 t_2 t_3 \ldots \text{ (octal)} .$$

It is evident that ϕ is strictly increasing and maps $[0,1]$ onto a subset of $[0, 1/7]$. (We choose to set $\phi(1) = .11111\ldots$ (octal) $= 1/7$.)

If $p = (x, y, z) \in I^3$, set $\phi(x) = a$, $2\phi(y) = b$, $4\phi(z) = c$ and write each in octal:

$$a = .a_1 a_2 a_3 a_4 \ldots \text{ (octal)}$$

$$b = .b_1 b_2 b_3 b_4 \ldots \text{ (octal)}$$

$$c = .c_1 c_2 c_3 c_4 \ldots \text{ (octal)} .$$

From the definition of ϕ, we see that each a_i is either 0 or 1, that each b_i is either 0 or 2, and each c_i either 0 or 4. With $h(x, y, z) = \phi(x) + 2\phi(y) + 4\phi(z) = a + b + c$, we see that

$$h(p) = .d_1 d_2 d_3 d_4 \ldots \text{ (octal)}$$

where each digit d_i is just $a_i + b_i + c_i$, no carrying being necessary except in the special case $x = y = z = 1$. Furthermore, since each integer in $\{0, 1, 2, \ldots, 7\}$ has a unique expression as sum of three selections from the set $\{0, 1, 2, 4\}$, distinct points $p = (x, y, z)$ yield distinct values $h(p)$.

We now turn to the class $\mathfrak{F}_0$ of functions that are continuously representable, and the weaker class $\mathfrak{F}_w$, and ask for criteria that distinguish their members from other functions. Referring back to the mapping diagram (16) we note that any such characteristic properties of a representable function $F \in \mathfrak{F}_w$ must arise from the special nature of the continuous functions h. The functions $h(x, y) = \phi(x) + \psi(y)$ defined on a rectangle $S = I \times J \subset R^2$ form a proper closed subspace $\mathfrak{V}$ of $C[S]$. If P_1, P_2, P_3, P_4 are the successive vertices of a rectangle that lies in S, then the alternating sum of point masses at the points P_i is a functional that annihilates $\mathfrak{V}$. If $h \varepsilon \mathfrak{V}$, then

$$h(P_1) - h(P_2) + h(P_3) - h(P_4) = 0 \quad . \tag{17}$$

Conversely, if $h \varepsilon C[S]$ and if (17) holds for all choices of the points P_i, then $h \varepsilon \mathfrak{V}$. This property is readily extended to any chain of $2n$ points P_i which are vertices of a closed polygon in S with edges that are successively vertical and horizontal. (See Figure 3 and [5].)

We next examine the level sets of functions $h \varepsilon \mathfrak{V}$. Any set that is a subset of some particular level set for a function g will be called a sublevel set for g; thus, a λ-sublevel set for g is a set on which g is constantly λ. The $\mathfrak{V}$ sets in S will be the collection of all sets that are sublevel sets for any of the functions $h \varepsilon \mathfrak{V}$. That some of these

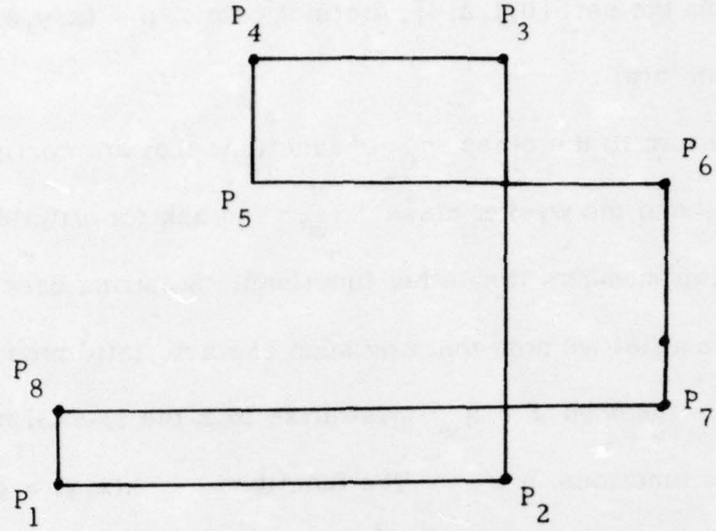

Figure 3

sets are complicated in structure can be seen by considering an h for which the component functions ϕ and ψ are continuous but everywhere non-differentiable.

Suppose that $F \varepsilon \mathfrak{F}_w(S)$, with $F = f \circ h$. If E is a sublevel set for h in S, then E is a sublevel set for F. Our next result is a simple converse of this which turns out to be quite useful; because of it, any property that is common to all the $\mathfrak{V}$ sets must also hold for certain (and sometimes all) of the level sets of a representable function F.

<u>Theorem</u> 9. <u>Let</u> $F \varepsilon \mathfrak{F}_w(S)$. <u>Then, thin connected sublevel sets for</u> F <u>in</u> S <u>must be</u> $\mathfrak{V}$ <u>sets. Specifically, if</u> $F = f \circ h$ <u>on</u> $S = I \times J$, <u>and</u> E_λ <u>is the</u> λ<u>-level set for</u> F <u>in</u> S, <u>and</u> E <u>is any subset of</u> E_λ <u>that</u>

is connected and contains no interior point of E_λ, then E is a sublevel set for h.

Proof: Let $h(E) = \sigma \subset R$. If σ is a single point, E is a sublevel set for h. Suppose σ is an interval. Since F is constant on E, $f(t) = \lambda$ for all $t \varepsilon \sigma$. Choose $p_0 \varepsilon E$ so that $h(p_0)$ is interior to σ, and then a neighborhood $\mathfrak{G}$ about p_0 so that $h(\mathfrak{G}) \subset \sigma$, using the continuity of h. Clearly, we would then have $F(p) = \lambda$ for all $p \varepsilon \mathfrak{G}$, and p_0 would have been interior to E_λ.

We remark that if F is continuous, the set E could be taken as any component of the boundary in S of the set of points p with $F(p) > \lambda$.

A trivial illustration may be helpful. Suppose that a function $h(x, y) = \phi(x) + \psi(y)$ is constant on a vertical segment α in $S = I \times J$. Clearly, ψ is constant on a subinterval $J_0 \subset J$, and h will then also be constant on every segment in S parallel to α. By virtue of Theorem 9, if $F \varepsilon \mathfrak{F}_w(S)$ and is not constant on any open set in S, then F must share this property; if F is constant on vertical segment, it must be constant on each parallel segment. This shows immediately that functions such as $(x - c)^2 e^y$ do not belong to the weak class $\mathfrak{F}_w(S)$ on any open rectangle S that contains a portion of the line $x = c$. (In passing, we note that the functions $(x - c)^2 G(y)$, with G of class C^2, satisfy the differential equation (15) everywhere.)

Arnol'd noted a special case of this in [2] in connection with the function $F(x, y) = xy$, which --- for the same reason --- does not belong to the class $\mathfrak{F}_W$ on any open set containing the origin. However, in the open first quadrant, we note that this function is $\exp(\log x + \log y)$, which belongs to $\mathfrak{F}_\infty$. This same example shows that functions in the class $\mathfrak{F}_\infty$ can converge uniformly to functions that are not even in $\mathfrak{F}_W$. For example,

$$F_n(x, y) = (x^2 + \frac{1}{n})\, e^y$$

belongs to $\mathfrak{F}_\infty(R^2)$, and converges uniformly on all compact sets to $x^2 e^y$ which does not belong to $\mathfrak{F}_W$ on any open set containing a point of the Y axis.

To apply Theorem 9 further, we need other characteristic properties of $\mathbf{V}$ sets. We can obtain one by using the annihilating property of the linear functionals given by (17), also used by Arnol'd in a similar fashion in [2].

<u>Lemma</u> 3. <u>If</u> E <u>is a level set for</u> $h \varepsilon \mathbf{V}$ <u>in the rectangle</u> $S = I \times J$, <u>and</u> E <u>contains any three of the four points</u> (a_i, b_j), $i, j = 0, 1$, <u>then</u> E <u>contains the fourth.</u>

Proof: If $P_{i+2j} = (a_i, b_j)$, then

$$h(P_0) - h(P_1) + h(P_3) - h(P_2) = 0$$

so that if, for example, $h(P_0) = h(P_1) = h(P_2) = \lambda$, then $h(P_3) = \lambda$.

Combining this with Theorem 9, we have:

<u>Theorem</u> 10. <u>If</u> $S = I \times J$, <u>and</u> $F \varepsilon \mathfrak{F}_W(S)$, <u>if</u> E_λ <u>is the</u> λ<u>-level set</u>

for F *in* S, *and if* E *is a connected subset of* E_λ *containing no interior point of* E_λ *but which contains three of the points* (a_i, b_j), $i, j = 0, 1$, *then* E_λ *must contain the fourth point*.

For example, $F(x, y) = x^2 + xy + y^2$ does not belong to the class $\mathfrak{F}_w$ on the square $|x| < 2$, $|y| < 2$ since it has thin level sets in the form of tilted ellipses that do not have the four-point property described in Lemma 3, and Theorem 10. This same argument can be applied to any of the functions $x^2 + \beta xy + y^2$ with $\beta^2 \neq 4$, and to any neighborhood of the origin, thus showing that none of these functions is locally weakly nomographic at $(0, 0)$. However, the present method does not answer the question of local nomographic representability for these functions at any point other than the origin, since the four-point property is a global property of a single level line, and at no other point than $(0, 0)$ do the level lines of the function $x^2 + xy + y^2$ converge. [We note in passing that these functions, as $\beta \to 2$, provide an example of functions lying outside the class $\mathfrak{F}_W$ but which converge uniformly on compact sets to a function in $\mathfrak{F}_\infty$, namely $(x + y)^2$.]

We therefore need another property of the $\mathcal{V}$ sets that is more local in character. We note that such properties must have some refinement since the level lines of the function xy, which is not in $\mathfrak{F}_W$, very closely resemble, locally, the level lines of the function $x^\delta + y^\delta$ when δ is very small.

6. The six-point construction

To obtain the desired property, we return to the class of functionals (17) which annihilate $\mathcal{V}$, and consider one based on six point measures. Suppose that six points p_i are located as indicated in Figure 4, forming vertices of a closed polygon with edges alternatingly vertical and horizontal. Then, for any function $h \in \mathcal{V}$, one has

$$(18) \qquad h(p_1) - h(p_2) + h(p_3) - h(p_4) + h(p_5) - h(p_6) = 0 .$$

Suppose that $h(p_1) = h(p_4)$ and $h(p_3) = h(p_6)$. Then it must follow that $h(p_2) = h(p_5)$.

This means that, given two points on distinct level lines of a function h in $\mathcal{V}$, a geometric construction will produce pairs of points that must lie on a third level line. We illustrate this procedure in an especially simple case. Suppose that we are dealing with a specific function $h \in \mathcal{V}$, and we choose two points P_1 and P_2 with $h(P_1) \neq h(P_2)$. Suppose that the level lines of h containing these points are as shown in Figure 5. If we construct the dotted lines in the figure, thereby locating the additional points Q_1, Q_2, P_3, Q_3, we see that it is necessarily true that $h(P_3) = h(Q_3)$.

Of course, not all functions h will have such simple level sets --- e.g., $h(x, y) = (x \sin x^{-1})^2 + (y \sin y^{-1})^2$. However, we can show that the six-point construction described above will always work, locally, for functions h that obey certain convenient restrictions.

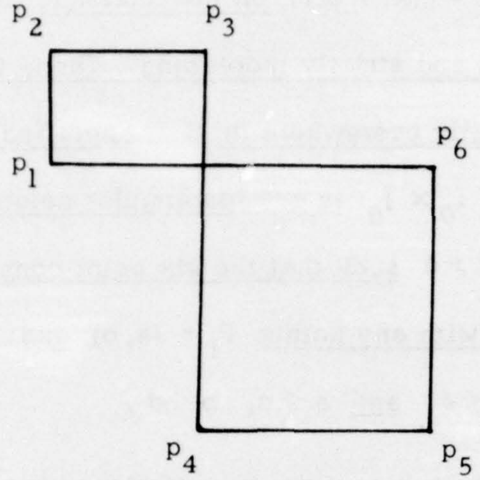

Figure 4

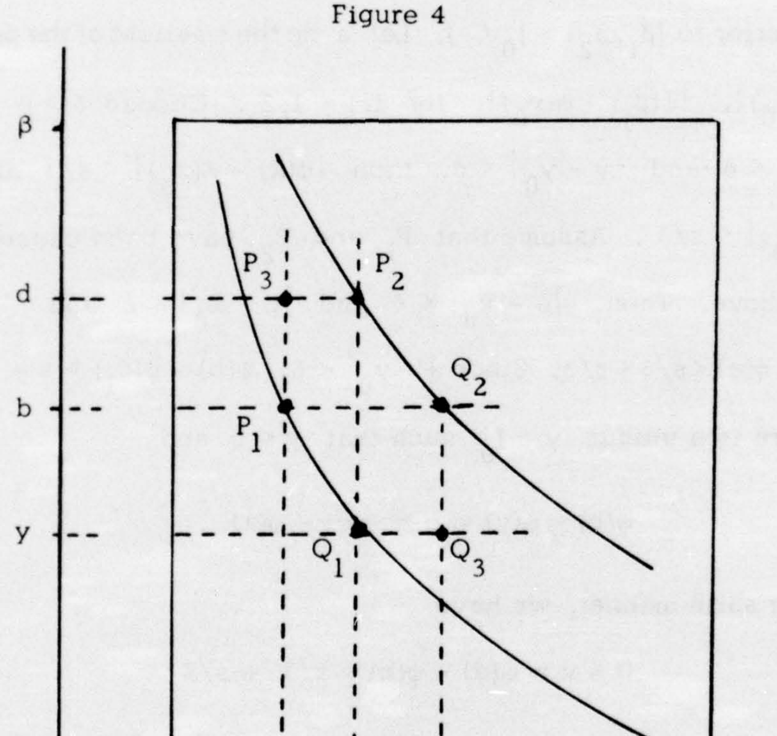

Figure 5

Theorem 11. Let $h(x, y) = \phi(x) + \psi(y)$ on the rectangle $S = I \times J$, where ϕ and ψ are continuous and strictly increasing. Then, the six-point construction applies locally everywhere in S. Specifically, if p_0 is interior to S, and $S_0 = I_0 \times J_0$ is any rectangular neighborhood of p_0 in S, then there is a $\delta > 0$ such that the six point construction can be applied in S_0, starting with any points $P_1 = (a, b)$ and $P_2 = (c, d)$ in S_0 such that $|P_i - p_0| < \delta$, and $a < c$, $b < d$.

Proof: Let $p_0 = (x_0, y_0)$, and assume x_0 interior to the interval $[\alpha_1, \alpha_2] = I_0 \subset I$, and y_0 interior to $[\beta_1, \beta_2] = J_0 \subset J$. Let s be the smallest of the positive numbers $|\phi(\alpha_i) - \phi(x_0)|$, $|\psi(\beta_j) - \psi(y_0)|$, for $i, j = 1, 2$. Choose $\delta > 0$ so that if $|x - x_0| < \delta$ and $|y - y_0| < \delta$, then $|\phi(x) - \phi(x_0)| < s/3$ and $|\psi(y) - \psi(y_0)| < s/3$. Assume that P_1 and P_2 have been chosen as described above. Then, $|a - x_0| < \delta$ and $|c - x_0| < \delta$ and $0 < u = \phi(c) - \phi(a) < s/3 + s/3$. Since $|b - y_0| < \delta$, $\psi(b) - \psi(\beta_1) \geqq s - s/3 > u$. Hence, there is a unique $y \in J_0$ such that $y < b$ and

$$\psi(b) - \psi(y) = u = \phi(c) - \phi(a) \quad . \tag{19}$$

In the same manner, we have

$$0 < v = \psi(d) - \psi(b) < s/3 + s/3$$

while

$$\phi(\alpha_2) - \phi(c) \geqq s - s/3 > v \quad .$$

Hence, there is a unique $x \in I_0$ such that

$$\phi(x) - \phi(c) = v = \psi(d) - \psi(b) \tag{20}$$

Finally, we set $Q_1 = (c, y)$, $Q_2 = (x, b)$, $P_3 = (a, d)$ and $Q_3 = (x, y)$. All lie in S_0, and by (19) $h(P_1) = h(Q_1)$, and by (20), $h(P_2) = h(Q_2)$. These in turn imply that $h(P_3) = h(Q_3)$. [A similar result holds if ϕ is increasing, and ψ decreasing].

The reason for considering this special case lies in the following simple result, which permits us to use Theorem 11 for an interesting class of functions $F(x, y)$, those that are separately univalent.

<u>Lemma</u> 4. <u>Let</u> F <u>be continuous on a rectangle</u> $S = I \times J$, <u>and</u> (a, b) <u>an interior point of</u> S <u>such that</u>

(21) <u>if</u> $x_i \varepsilon I$ <u>and</u> $F(x_1, b) = F(x_2, b)$, <u>then</u> $x_1 = x_2$

(22) <u>if</u> $y_i \varepsilon J$ <u>and</u> $F(a, y_1) = F(a, y_2)$, <u>then</u> $y_1 = y_2$.

<u>If</u> $F \varepsilon \mathfrak{F}_w(S)$, <u>then</u> F <u>can be written as</u> $f \circ h$ <u>where</u> $h(x, y) = \phi(x) + \psi(y)$ <u>and</u> ϕ <u>and</u> ψ <u>are continuous and strictly monotonic on</u> I <u>and</u> J, <u>respectively, and</u> $\phi(a) = \psi(b) = 0$. <u>If, in addition,</u> $I = J$ <u>and</u> $F(x, y) = F(y, x)$, <u>then we can take</u> $\phi = \psi$, <u>with both increasing.</u>

Proof: If $\phi(x_1) = \phi(x_2)$, then $F(x_1, b) = f(\phi(x_1) + \psi(b)) = f(\phi(x_2) + \psi(b)) = F(b, x_2)$, so that $x_1 = x_2$. Since ϕ is continuous and univalent, ϕ is monotonic on I. The same argument applies to ψ. Set $\phi_0(x) = \phi(x) - \phi(a)$, $\psi_0(y) = \psi(y) - \psi(b)$, $f_0(t) = f(t + \varphi(a) + \psi(b))$, and we have $F = f_0 \circ h_0$ where $h_0(x, y) = \phi_0(x) + \psi_0(y)$. Suppose now that $I = J$ and F is symmetric. Since $F(a, x) = F(x, a)$, we can take $b = a$. Suppose that ϕ were increasing but ψ decreasing. We can then choose u and v with $u < a < v$ and $\phi(v) = \psi(u)$. However, we would then have $F(a,u) = F(v,a) = F(a,v)$, with $u \neq v$. We can therefore assume that ϕ and ψ are both increasing. Finally, suppose that there is $s \varepsilon I$ with $a < s$ and $\phi(s) < \psi(s)$. There must exist s_0, $a < s_0 < s$, with $\psi(s_0) = \phi(s)$. As before, $F(a,s) = F(s,a) = F(a,s_0)$, and $s = s_0$. Thus, $\phi = \psi$.

This yields a necessary condition for weak nomographic representability that is local and widely applicable.

Theorem 12. *Let* $F \varepsilon \mathfrak{F}_w(S)$ *where* $S = I \times J$, *and suppose that* F *is separately univalent on* S. *Then, for any* a *and* c *in* I *and* b *and* d *in* J, *with* $|a - c|$ *and* $|b - d|$ *sufficiently small, there must exist* x *and* y *near* a *and* b, *respectively, such that*

$$
\begin{aligned}
&F(a,b) = F(c,y) \\
(23) \qquad &F(c,d) = F(x,b) \\
&F(a,d) = F(x,y) \, .
\end{aligned}
$$

This results immediately from the corresponding conditions on the function h for which $F = f \circ h$. Note that the conclusion gives three simultaneous equations for the two unknowns x and y. The fact that these are not in general solvable for arbitrary F provides a method for proving that a specific function F is nowhere locally representable in the class $\mathfrak{F}_w$. Before illustrating this with examples, we append another generalization, dependent upon F being symmetric.

Theorem 13. Let $F \in \mathfrak{F}_w(S)$ where $S = I \times I$, and suppose that F is separately univalent on S, and that $F(x, y) = F(y, x)$. Then, for any a and b in I, $a < b$, there exist values of x_i in I, $a = x_0 < x_1 < x_2 < \ldots < x_m = b$, for any $m \geqq 3$, which solve the $m(m-1)/2$ equations

$$F(x_j, x_{i+1}) = F(x_i, x_{j+1}) \quad 0 \leqq i < j \leq m-1 \ . \tag{24}$$

Proof: As before, we can assume that $F = f \circ h$, where $h(x, y) = \phi(x) + \phi(y)$, and ϕ is strictly increasing on I. Set $d = \{\phi(b) - \phi(a)\}/m$, and then choose points $x_i \in I$ with $a = x_0 < x_1 < \ldots < x_{m-1} < x_m = b$ so that $\phi(x_{i+1}) - \phi(x_i) = d$. For any i and j with $i < j$, we then have

$$\phi(x_{i+1}) - \phi(x_i) = \phi(x_{j+1}) - \phi(x_j)$$

which implies $h(x_{i+1}, x_j) = h(x_i, x_{j+1})$, and thus (24). We note that (24) is a system of $m(m-1)/2$ simultaneous equations in $m-1$ unknowns. This imposes a severe restriction on any specific function F, and makes it possible to exclude some functions from membership in $\mathfrak{F}_w$. Since the condition applies to any square set S along the diagonal, this provides a criterion for local non-representability.

To illustrate the use of Theorems 12 and 13, consider the function $F(x, y) = x^2 + xy + y^2$. From Theorem 10, we saw that this is not locally weakly nomographic at $(0, 0)$. We now see that it is locally nomographic nowhere in the first quadrant. Specifically, one must show that if $a > 0$, $b > 0$, then there exist infinitely many c, d with $a < c$, $b < d$, such that the system

$$c^2 + cy + y^2 = a^2 + ab + b^2$$

$$x^2 + bx + b^2 = c^2 + cd + d^2 \tag{25}$$

$$x^2 + xy + y^2 = a^2 + ad + d^2$$

is inconsistent. Since this is (23), Theorem 12 applies.

If, instead, we use (24) --- since F is symmetric --- we may take $m = 4$, and then if we show that the system

$$3\,x_1^2 = a^2 + ax_2 + x_2^2$$

$$3\,x_2^2 = a^2 + ab + b^2$$

$$3\,x_3^2 = b^2 + bx_2 + x_2^2$$

$$x_1^2 + x_1x_2 + x_2^2 = a^2 + ax_3 + x_3^2$$

$$x_2^2 + x_2x_3 + x_3^2 = b^2 + bx_2 + x_2^2$$

$$x_1^2 + x_1x_3 + x_3^2 = a^2 + ab + b^2$$

does not have a solution for any choice of a and b with $0 < a < b$, we will also have proved that F is not locally nomographic.

Of these algebraic tasks, the first is easier, and therefore preferable. From (25), one derives the relation

$$(x - c)(y - b) + (c - a)(d - b) = 0 .$$

If $c = a + \alpha$ and $d = b + \beta$, then one finds that either $c = b$, $c = -b$, or

$$a(\beta + 2\alpha) + b(2\beta + \alpha) + \alpha^2 + \alpha\beta + \beta^2 = 0$$

none of which are acceptable since $0 < a < c$, $0 < b < d$, and (c, d) can be chosen arbitrarily in a neighborhood of (a, b) .

Unfortunately, each application of Theorem 12 or 13 to test a specific function seems ad hoc; the method is powerful but tedious to apply. For example, to show that $2x^2y + xy^2$ is nowhere locally nomographic in the first quadrant requires that one show inconsistency of the system

$$2x^2b + xb^2 = 2c^2d + cd^2$$

$$2c^2y + cy^2 = 2a^2b + ab^2$$

$$2x^2y + xy^2 = 2a^2d + ad^2$$

(excluding the inadmissible solutions such as $x = a$, $y = d$ with $a = c, b = d$.)

All of these support the following conjecture: a polynomial $F(x, y)$ will not belong to the weak class $\mathfrak{F}_w$ on any open set unless it satisfies the differential equation (15), and can be written as $f(\phi(x) + \psi(y))$ with f, ϕ and ψ polynomials.

7. The closure of the nomographic functions

The purpose of this section is to obtain characteristic properties of functions in the closure of the weak class $\mathfrak{F}_w(S)$ where S is a fixed rectangle. This will then provide a method for showing that specific functions G cannot be approximated uniformly on compact sets by nomographic functions $f(\phi(x) + \psi(y))$ where ϕ and ψ are continuous, but f unrestricted. Our proofs require that the function G be such that $G_xG_y \neq 0$ on S, which we take to be $I \times J$ where $I = [a, b]$, $J = [c, d]$. For convenience, we deal here only with the case in which $G_x > 0$ and

$G_y > 0$ on S. It then follows that G is separately strictly monotonic in S, and that there is a constant M such that if p and q are points of S with $p \ll q$, (meaning that $p = (x_1, y_1)$, $q = (x_2, y_2)$ and $x_1 \leqq x_2$, $y_1 \leqq y_2$) and $|G(p) - G(q)| < \delta$, then $|p - q| < M\delta$.

If G lies in the closure of $\mathfrak{F}_w(S)$, then for any $\epsilon > 0$ there exists $F \in \mathfrak{F}_w(S)$ such that $\|G - F\| < \epsilon/2$. F will then inherit some of the properties of G in a modified form. One can expect certain similarities between the level lines of F and of G so that the six-point construction might be applied again. We first observe that if p and q lie in S with $p \ll q$, and if $F(p) = F(q)$, then $|G(p) - G(q)| < \epsilon$ and $|p - q| < M\epsilon$. Likewise, if $p \ll q$ and $h(p) = h(q)$, then $F(p) = F(q)$, and again, $|p - q| < M\epsilon$. Accordingly, the component functions ϕ and ψ that enter into h must have a weak form of univalence. Suppose that x_1 and x_2 lie in I and $\phi(x_1) = \phi(x_2)$. We may assume that $x_1 < x_2$. Then, $(x_1, c) \ll (x_2, c)$, and $h(x_1, c) = h(x_2, c)$, so that $|x_1 - x_2| < M\epsilon$. The same argument applies to ψ on J. This suggests a useful definition.

A continuous real function g on $[\alpha, \beta]$ is said to be quasi-monotonic (q. m.) with gap Δ if $g(x) = g(y)$ implies $|x - y| < \Delta$, for any x, y in $[\alpha, \beta]$.

Observe that we have just shown that ϕ and ψ are quasi-monotonic with gap $\Delta = M\epsilon$. Since ϵ can be arbitrarily small, we henceforth assume that $\Delta < L/6$, where L is the shortest side of S.

Quasi-monotonic functions are almost monotonic on their domain. They need not be locally 1-to-1, nor do they have to be uniformly close to a monotonic function. However, they have the following property.

<u>Lemma 5</u>. <u>If</u> g <u>is</u> q. m. <u>with gap</u> Δ <u>on</u> $[\alpha, \beta]$, <u>and</u> $g(\alpha) < g(\beta)$, <u>and</u> $\alpha \leqq x < y \leqq \beta$ <u>with</u> $y - x > \Delta$, <u>then</u> $g(x) < g(y)$. <u>In particular, if</u> $\alpha + \Delta < x < \beta - \Delta$, <u>then</u> $g(\alpha) < g(x) < g(\beta)$.

Proof: If $g(x) > g(y)$, then there exists z, $\alpha \leqq z \leqq x$ with $g(z) = g(y)$. Clearly, $|z - y| > \Delta$, which is impossible. The last statement in the lemma follows by specializing x and y.

Thus, when ϵ is sufficiently small and $F \in \mathfrak{F}_W(S)$ with $\|G - F\| < \epsilon/2$, we may assume that $F = f \circ h$ where $h(x, y) = \phi(x) + \psi(y)$ and ϕ and ψ are quasi-monotonic with gap Δ on the intervals I and J respectively. As before, we can also assume that this representation has been modified if necessary so that $\phi(a) = \psi(c) = 0$, and $A = \phi(b) \geqq 0$. Put $B = \psi(d)$.

<u>Lemma 6</u>. $A > 0$ <u>and</u> $B > 0$.

Proof: Examine the behavior of $h(x, y)$ on the boundary of S. At the lower left corner, (a, c), it takes the value 0, while $h(a, d) = B$ and $h(b, c) = A$. Since the distance from (a, c) to any point along the upper edge or the right hand edge of S is greater than Δ, and since all such points p obey $(a, c) \ll p$, it follows that h cannot take the value 0 anywhere on these edges. Since $A \geqq 0$, $A > 0$. If $B \leqq 0$, then there would have to be a point p on these edges where $h(p) = 0$; thus, $B > 0$.

We now show that the six-point construction applies to $h(x, y)$ in the rectangle S.

Theorem 14. *Let* (u,v) *be any point in* S *such that* $|2u-(a+b)| < 2L/3$, $|2v-(c+d)| < 2L/3$. *Then, one of the following statements must be true:*

(i) *there exist* x *and* y *in* J *such that* $h(a,x) = h(u,c)$,

$$h(a,y) = h(b,c), \quad h(b,x) = h(u,y)$$

(ii) *there exist* x *and* y *in* I *such that* $h(x,c) = h(a,v)$,

$$h(y,c) = h(a,d), \quad h(x,d) = h(y,v) \ .$$

Proof: The choice between (i) and (ii) depends on the comparative size of A and B. Suppose $A \leqq B$. Choose $y \varepsilon J$ so that $\psi(y) = A$. Since u is within $L/3$ of the midpoint of I, and since $\Delta < L/6$, $a + \Delta < u < b - \Delta$. Hence, $0 < \phi(u) < A$. Choose $x \varepsilon J$ so that $\psi(x) = \phi(u)$. Inspection then shows that the statements in (i) are all valid. If $B \leqq A$, a similar argument leads to (ii).

This in turn yields the desired criterion for approximate representability.

Theorem 15. *Let* G *be continuous on* $S = I \times J$, *where* $I = [a,b]$ *and* $J = [c,d]$. *Suppose also that* $G_x > \sigma$ *and* $G_y > \sigma$ *on* S, $\sigma > 0$. *Let* (u,v) *be any point in* S *such that* $|2u-(a+b)| < 2L/3$ *and* $|2v-(c+d)| < 2L/3$, *where* L *is the length of the shorter side of* S. *If* G *lies in the uniform closure of* $\mathfrak{F}_w(S)$ *then for any sufficiently small* ϵ $[\epsilon < L\sigma/12$ will do$]$, *one of the following statements must hold:*

(i) *there exist* x *and* y *in* J *such that*

$$|G(a,x) - G(u,c)| < \epsilon$$
$$|G(a,y) - G(b,c)| < \epsilon$$
$$|G(b,x) - G(u,y)| < \epsilon$$

(ii) <u>there exist</u> x <u>and</u> y <u>in</u> I <u>such that</u>

$$|G(x, c) - G(a, v)| < \epsilon$$
$$|G(y, c) - G(a, d)| < \epsilon$$
$$|G(x, d) - G(y, v)| < \epsilon$$

Proof: Choose ϵ small, and $F \varepsilon \mathfrak{F}_w(S)$ so that $\|G - F\| < \epsilon/2$. The previous analysis applies to F and h must satisfy one of the statements (i) and (ii) in Theorem 14. Since $F = f \circ h$, so must F, and since $|G(p) - F(p)| < \epsilon/2$ for any $p \varepsilon S$, the statements (i) and (ii) in Theorem 15 hold. The number M in the previous discussion can be taken to be $2/\sigma$, so that $L/(6M) = L\sigma/12$.

That this result is an effective criterion can be seen by a simple illustration. Let S *be the square with* (1, 0) and (2, 1) an opposite vertices, and take $(u, v) = (1.5, .5)$. Then, if $G(x, y) = x^2 + xy + y^2$, statement (ii) in Theorem 15 is the assertion that the following system has a solution for all sufficiently small ϵ:

$$(26) \qquad \begin{aligned} &|x^2 - 1.75| < \epsilon \\ &|y^2 - 3| < \epsilon \\ &|(x^2 + x + 1) - (y^2 + .5y + .25)| < \epsilon . \end{aligned}$$

However, if $x = \sqrt{1.75}$ and $y = \sqrt{3}$, the third inequality becomes $|4.07287 - 4.116025| < \epsilon$, and (26) cannot have a simultaneous solution for small ϵ. Hence, $x^2 + xy + y^2$ cannot be approximated uniformly on S by functions in the weak nomographic class $\mathfrak{F}_w$. In the same way, using small rectangles S, one may verify that $x^2 + xy + y^2$ is nowhere locally approximable by functions from $\mathfrak{F}_W$ anywhere in the open first quadrant.

Finally, we note that this same approach enables one to estimate the uniform distance from a function G to the class $\mathfrak{F}_W$. For example, from (26), construct the function

$$K(x, y) = |x^2 - 1.75| + |y^2 - 3| + |x^2 + x + 1 - y^2 - .5y - .25| .$$

Since K is never 0 on S, it must have a positive minimum, γ. However, by the argument used in Theorem 15, if $\|G - F\| < \epsilon/2$ for some $F \in \mathfrak{F}_W(S)$, then (26) will have solutions x and y for which $K(x, y) < 3\epsilon$. Thus, $\epsilon > \gamma/3$, and we have obtained a lower bound on the distance from $x^2 + xy + y^2$ to $\mathfrak{F}_W(S)$.

When $I = J$ and G is also symmetric, so that $G(x, y) = G(y, x)$, another approach can also be used, similar to that of Theorem 13. As before, suppose that $\|G - F\| < \epsilon/2$ where $F = f \circ h$ and $h(x, y) = \phi(x) + \psi(y)$ on $S = [a, b]^2$, and ϕ and ψ are quasi-monotonic and $\phi(a) = \psi(a) = 0$, $\phi(b) = A$, $\psi(b) = B$. Even though G is symmetric, it does not follow that $\phi = \psi$. However, if $\phi(x) = \psi(y)$, then $h(a, y) = h(x, a)$ and $F(a, y) = F(x, a)$, so that $|G(a, y) - G(x, a)| < \epsilon$. But, G is symmetric, so $|G(a, y) - G(a, x)| < \epsilon$ and $|x - y| < M\epsilon = \Delta$. In particular, it follows that if $A \leqq B$, there is a $\bar{y}$ in I with $\bar{y} > b - \Delta$ and $\psi(\bar{y}) = A$, while if $B \leqq A$, there is an $\bar{x}$ in I with $\bar{x} > b - \Delta$ and $\phi(\bar{x}) = B$. Assume that we have $A \leqq B$; then, for any integer $m \geqq 3$, we can choose x_i

and y_j in I with $a = x_0 < x_1 < x_2 < \ldots < x_m = b$ and $a = y_0 < y_1 < \ldots < y_m = \bar{y}$ so that $\phi(x_{i+1}) - \phi(x_i) = A/m = \psi(y_{j+1}) - \psi(y_j)$. This implies that $F(x_i, y_{j+1}) = F(x_{i+1}, y_j)$ for all i and j, and thus $|G(x_i, y_{j+1}) - G(y_j, x_{i+1})| < \epsilon$. We have thus proved the following.

Theorem 16. Let G be continuous and symmetric on $S = [a, b]^2$, and suppose that G_x and G_y are bounded below by $\sigma > 0$ on S. Then, for any $\delta > 0$ and any sufficiently small ϵ, and any integer $m \geqq 3$, there exist points x_i and y_j in I such that $a = x_0 < x_1 < \ldots$, $a = y_0 < y_1 < y_2 < \ldots$, with $x_m > b - \delta$ and $y_m > b - \delta$, and

$$|G(x_i, y_{j+1}) - G(y_j, x_{i+1})| < \epsilon$$

for all i and j.

Most of the results in this and the two preceding sections carry over to the class of functions of n-variables of the form $F(x_1, x_2, \ldots, x_n) = f(\phi_1(x_1) + \phi_2(x_2) + \ldots + \phi_n(x_n))$ since they depend mainly on the class of annihilating functional for the class $\mathfrak{X}$, which has an immediate analogue in the more general case.

8. Other classes and methods

We once more change our notation so that $\mathfrak{F}_n(S)$ now refers to the class of those functions $F(x, y, z)$ which can be represented on the set S in the form $f(\varphi(x, y), \psi(y, z))$, with f, φ and ψ of class C^n. As seen in Section 2, any function F has such a representation if f, φ and ψ are unrestricted. However, it is natural to conjecture that $\mathfrak{F}_0(S)$ is a relatively thin subset of $C[S]$, and that a function such as $F_0(x, y, z) = xy + yz + xz$ does not even belong to the closure of $\mathfrak{F}_w(S)$ where S is any cube in the open positive octant. We are not able to prove this much; the class $\mathfrak{F}_0(S)$ is considerably more complex in structure than those discussed in previous sections, and much less tractable. The following elementary result illustrates this fact.

<u>Theorem 17</u>. <u>Any function</u> F <u>in</u> $\mathfrak{F}_4(S)$ <u>is the solution in</u> S <u>of a specific</u> 4th <u>order partial differential equation with</u> 55 <u>terms</u>.

Proof: Assuming that $F(x, y, z) = f(\varphi(x, y), \psi(y, z))$, we have $F_x = f_1\varphi_1$, $F_y = f_1\varphi_2 + f_2\psi_1$, $F_z = f_2\psi_2$, using a subscript notation for partial derivatives. Hence,

$$F_y = F_x G + F_z H \tag{27}$$

where G is independent of z and H is independent of x. Differentiating again, we obtain

$$F_{xy} = F_{xx} G + F_{xz} H + F_x G_1$$

$$F_{yz} = F_{xz} G + F_{zz} H + F_z H_2$$

$$F_{xyz} = F_{xxz} G + F_{xzz} H + F_{xz} G_1 + F_{xz} H_2 .$$

Together with (27), we now have four equations in G, H, G_1, H_2 which can be solved, obtaining an expression for G solely in terms of derivatives of G. Since $\partial G/\partial z \equiv 0$, we can differentiate this expression and thus obtain the desired equation which must be satisfied by F. (Since it is a long and tedious calculation and the result contains so many terms, there is little reason to include the final equation here. Should any reader of this be sufficiently curious to wish to see the complete equation, it may be obtained by writing the author.)

In spite of this inherent complexity, it is possible to prove a non-representability theorem, assuming only C'.

<u>Theorem 18</u>. <u>The function</u> $F(x, y, z) = xy + yz + xz$ <u>does not belong to</u> $\mathfrak{F}_1(\mathbb{R}^3)$.

Proof: We assume that $F = f(\varphi(x, y), \psi(y, z))$, with f, φ, and ψ differentiable, and obtain

$$F_x = f_1\varphi_1 = y + z$$

$$F_y = f_1\varphi_2 + f_2\psi_1 = x + z$$

$$F_z = f_2\psi_2 = y + x \ .$$

Then put $G(x, y) = \varphi_2(x, y)/\varphi_1(x, y)$, and $H(y, z) = \psi_1(y, z)/\psi_2(y, z)$ to obtain

$$x + z = (y + z)\, G(x, y) + (y + x)\, H(y, z) \ . \tag{28}$$

If we solve this for G we obtain

$$G(x, y) = \frac{1 - H(y, z)}{y+z}\, x + \frac{z - yH(y, z)}{y+z} \ .$$

Since the left side is independent of z for each x, and the right side has the form $\alpha(y, z)x + \beta(y, z)$, it follows that $\alpha(y, z)$ and $\beta(y, z)$ are each independent of z, giving us the relation

$$G(x, y) = \alpha(y)x + \beta(y) \ . \tag{29}$$

Similarly, we can solve (28) for H to obtain

$$H(y,z) = \frac{1-G(x,y)}{x+y} z + \frac{x-yG(x,y)}{x+y}$$

and by a similar argument, find that

$$H(y,z) = \gamma(y)z + \delta(y) . \tag{30}$$

Putting (29) and (30) back into (28), we have

$$\begin{aligned} x + y = &\{y\,\alpha(y) + \delta(y)\}x + \{y\,\gamma(y) + \beta(y)\}z \\ &+ \{\alpha(y) + \gamma(y)\}xz + \{\beta(y) + \delta(y)\}y \ . \end{aligned}$$

Since this holds for all x, y and z, it follows that

$$1 = y\,\alpha(y) + \delta(y)$$

$$1 = y\,\gamma(y) + \beta(y)$$

$$0 = \alpha(y) + \gamma(y)$$

$$0 = \beta(y) + \delta(y)$$

which are readily seen to be inconsistent. Accordingly, the assumed representation of $xy + yz + xz$ cannot hold.

It would obviously be of interest to prove that $xy + yz + xz$ is not a member of $\mathfrak{F}_0(\mathbb{R}^3)$, or of $\mathfrak{F}_w(\mathbb{R}^3)$. More usefully, it would be interesting to obtain generally applicable criteria for non-representability, and to investigate the approximation properties of these classes.

The results of the preceding sections use techniques that are ad hoc. There are certain common threads which may lead eventually to a general theory: (i) the analysis of level sets, (ii) the use of functional equations and inequalities, (iii) the determination of characteristic automorphisms and (iv) the use of the associated differential equations.

The general problem we are dealing with is that of finding significant characterizations for the class of functions that are representable by a specified format, or that may be uniformly approximated by such functions. The first technique, that of analyzing level sets, depends upon the recognition that we are dealing with a conventional factoring problem. Depending upon the exact nature of the superposition schemata under study, we have a mapping diagram such as the following

$$\begin{tikzcd} S \arrow[r, "k"] \arrow[drrr, "F"'] & A \arrow[r, "h"] & B \arrow[r, "g"] & C \arrow[d, "f"] \\ & & & R \end{tikzcd}$$

where F is given, and where we are attempting to decide if there are maps f, g, h, k lying in certain specific classes of mappings such that the diagram commutes. Clearly, if the nature of any of the component maps f, g, h, k force two points of S to have the same image in R, this behavior must be imitated by F, either exactly if we are dealing with membership or approximately if we are examining the uniform closure of a function class.

The method of functional equations and inequalities offers promise; it is well illustrated by the material in Section 6. Here, the nature of the functional format implied the existence of solutions to certain systems of equations, thus providing necessary conditions for representability.

However, at present it would seem that each schemata must be analyzed independently, and no general methods are in sight.

The third method is mathematically attractive, but as yet has achieved no success. One first seeks to characterize a given class of functions $\mathfrak{F}$ by the transformations that leave it invariant. For example, the class $\mathfrak{F}(R^2)$ of all nomographic functions F of the form $f(\varphi(x) + \psi(y))$ is carried into itself by the following mappings:

$$F \rightarrow G \quad \text{where} \quad G(x, y) = g(F(x, y))$$

$$F \rightarrow G \quad \text{where} \quad G(x, y) = F(y, x)$$

$$F \rightarrow G \quad \text{where} \quad G(x, y) = F(g(x), y)$$

If in addition, we specify that $\mathfrak{F}$ contains the function $x + y$, then we have uniquely characterized the class $\mathfrak{F}$. Does this help to settle such questions as the relative sparsity of $\mathfrak{F}$ as a subset of $C[\mathfrak{B}]$? Is there a similar characterization for the closure of $\mathfrak{F}$? Finally, does this in any way permit one to decide if an individual function F_0 lies in $\mathfrak{F}$, and if not, to determine the distance from F_0 to $\mathfrak{F}$, or to identify the nearest member of $\mathfrak{F}$?

The last general method deserves somewhat more discussion. As has been shown in several cases, it is generally possible to construct one or more partial differential equations which must be satisfied by any function F in the function class $\mathfrak{F}_\infty$. (See [17]) The reasons for this are in part combinatorial.

Theorem 19. If F is a C^{∞} function of N variables, then in general it has $\binom{N+k-1}{k}$ essentially different partial derivatives of order k.

Proof: Let the number in question be $A(N,k)$, so that we have $A(N,1) = N$ and $A(1,k) = 1$. $A(N,k)$ is also the number of sequences $c = \langle c_1, c_2, \ldots, c_k \rangle$ where $c_1 \leq c_2 \leq \ldots \leq c_k$ and $c_i \in \{1, 2, \ldots, N\}$. Partition the class $\mathcal{C}$ of such sequences c according to the value of c_1 so that $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \ldots \cup \mathcal{C}_N$. When $c_1 = 1$, the sequence $c' = \langle c_2, c_3, \ldots, c_k \rangle$ is $k-1$ long and its terms also lie in $\{1, 2, \ldots, N\}$. Hence, $\mathcal{C}_1$ contains $A(N, k-1)$ members. Similarly, if $c_1 = 2$, c' is a $k-1$ sequence with terms from $\{2, 3, \ldots, N\}$ so that $\mathcal{C}_2$ contains $A(N-1, k-1)$ members. This leads to the recursive equation

$$A(N,k) = \sum_{j=1}^{N} A(j,\ k-1)$$

from which the formula $A(N,k) = \binom{N+k-1}{k}$ is found in the usual way.

Corollary. If F is a C^{∞} function of N variables, then in general the number of essentially different partial derivatives of all orders k for $1 \leq k \leq m$ is given by

$$B(N,m) = \binom{N+m}{m} - 1 .$$

Suppose now that we are studying a specific function class, for example, those functions of three variables of the form

$$F(x,y,z) = f(\phi(x,y),\ \psi(y,z)) . \tag{31}$$

If we differentiate this m times, we will obtain $B(3,m)$ equations, one for each of the partial derivatives of F. Each of these equations will involve derivatives of f, φ and ψ of orders at most m. Since these three functions are functions of only two variables, the total number of new functions to arise will be at most $3B(2, m)$. We now ask if there is a choice of m such that

$$B(3, m) > 3B(2, m) \ .$$

We find that $B(3, 6) = 83$ and $3B(2, 6) = 81$.

Thus, if we were to differentiate (31) six times, we would have 83 equations involving at most 81 different derivatives of f, φ, and ψ. Eliminating these, we would be able to determine a single relationship involving all or some of the 83 different partial derivatives of F. (In fact, as indicated in Section 4, one need only differentiate four times, and the resulting monstrous nonlinear differential equation can be given explicitly.)

This is illustrative of the general case. Given a specific format for expressing a function F of N variables as a superposition of r functions of fewer variables, one may choose a sufficiently large m so that $B(N, m) > r\,B(N-1, m)$, and thus (in theory) arrive at a differential equation for F.

Sometimes, indeed, more then one equation must be satisfied. For example, if we are interested in functions of the form $F(x, y, z, w) = f(\varphi(x, y), \ \psi(z, w))$, then the general method outlined above shows that there ought to be a differential relation involving at most third order derivatives of F. In fact, it is easily seen that F must obey the following system of four second order equations

$$F_y F_{xz} - F_x F_{yz} = 0$$

$$F_y F_{xw} - F_x F_{yw} = 0$$

$$F_z F_{xw} - F_w F_{xz} = 0$$

$$F_z F_{yw} - F_w F_{yz} = 0 .$$

As the examples have shown, the differential equations one obtains are nonlinear and often of high degree, and in themselves unenlightening. Moreover, it is only for functions F that have very smooth representations that we can assert that the equations are satisfied. Is there any way that these equations can be used to settle questions of continuous representability? For example, we conjecture that any C^∞ function F which is of the form $f(\varphi(x,y), \psi(y,z))$ with f, φ and ψ continuous must in fact satisfy the characteristic 4th order differential equation for this family, almost everywhere: One method to approach this would be to prove that a C^∞ function which is the uniform limit of C^∞ solutions of the differential equation must itself be a solution. However, known results in the theory of nonlinear equations do not as yet support this.

We also conjecture that the distance from a function G to a function class $\mathfrak{F}$ on a compact set S can be estimated from below by means of the maximum of the corresponding differential expression, applied to G on the set S.

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Results are obtained dealing with the exact and the approximate representation of a function F as a superposition, in designated formats, of functions of fewer variables. Two main cases are considered. In the classical nomographic case, discussed in Sections 5, 6, 7, one seeks criteria for deciding if a function can be expressed in the form $f(\phi(x) + \psi(y))$, or as a uniform

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20. ABSTRACT (Cont'd.)

limit of such functions. The second case, discussed in Sections 2, 3, 4, is also related to the solution of Hilbert's 13th problem, and deals with the format $F(x) = f(\phi(x))$ (phi) where x lies in an n-cell I and ϕ (phi) is a real valued continuous function on I, and f is a function on R taking values in a chosen normed space $\mathcal{E}$ (epsilon). The use of these criteria is illustrated with several specific functions.

Since each format is associated with a specific partial differential equation, the results raise questions about the nature of the uniform closure of the C^{∞} solutions of such equations. Section 3 may also have more general interest since it shows that every continuous real function on an n-cell must share a certain universal property related to the metric dispersion of its level sets.